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Jet-Flapped Wings in Very Close Proximity to the Ground

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The method of matched asymptotic expansions is applied to the problem of jet-flapped wings of finite span in very close proximity to the ground. For the linearization of this problem, the order of small parameters, the angle of attack and jet deflection, are assumed to be smaller than the ratio of the ground clearance to the root chord. This linearized problem is solved as a direct problem represented by a source distribution on the upper surface of the wing and jet sheet with concentrated sources around the leading and side edges plus a separate confined channel flow region under the wing and jet sheet. The two-dimensional, jet-flapped airfoil is examined in detail, and the calculated lift coefficients lie within 5% of corresponding results by Lissaman.[1] In the three-dimensional case, a simple analytic solution is obtained for a flat plate semi-elliptic wing with a straight trailing edge, zero angle of attack and uniform momentum distribution of the jet. Spanwise lift distributions and lift coefficients are derived and the distributions of the jet momentum are discussed for minimum induced drag.

Introduction

RECENTLY, the possibility of using aerodynamics to support a high-speed ground transportation vehicle gives rise to an interesting class of problems in which a lifting surface translates in close proximity to a solid boundary.[2,3] The problem of airfoil section in ground proximity without jet flap (ram airfoils) has already been the object of several studies. In Ref. 4, the potential flow problem for two-dimensional flat plates in ground proximity is solved exactly by conformal transformations. Recently, three-dimensional problems have been solved by lifting surface theory with solutions generally requiring a high-speed computer.[5]

A jet flap is formed by a high-speed jet which emerges from a wing through a thin slit along its trailing edge. It affects the lift both by direct momentum reaction on the internal ducting and by changing the pressure distribution on the outer surface of the wing (i.e., supercirculation). Since this can increase the lift while most of the jet thrust is recovered as propulsion, the jet flap represents a way of combining the lifting and propulsive systems of jet aircraft. Early theoretical work for a two-dimensional wing in ground proximity with jet flap was done by Huggett[6] and others using the principle of a simple lifting line. Since then, Lissaman and the present authors gave an extension (by conformal transformation) to this ground effect problem,[1,7] and we applied the principle of the equivalence between the jet flap and the mechanical flap to this problem.[8]

All of these approaches are quite reasonable for wings which are not too close to the ground, such as might be encountered with an airfoil during takeoff and landing. However, when a lifting surface is designed to take advantage of ground effect the greatest interest lies in very close proximity. The theoretical work without jet flap for this case was done by Widnall and Barrows,[2,3] and the two-dimensional wing both with and without jet flap was studied by Cooke[9] using the principle of small perturbation technique. In the latter work, the basic physical limitation occurs when the jet curtain impinges on the ground and the numerical calculations for the jet-flapped airfoil have only been made for the half ground clearance to the chord, for the numerical methods are more complicated as the ground is approached.

The expected range of operation of high-speed ground transportation vehicles is such that the ratio of the ground clearance of the root chord is less than 0.1.[2,10] Also, while theoretical analysis of three-dimensional, jet-flapped wings have been developed by Maskell and Spence,[11] the effect of ground proximity has not been considered. In the present paper, the linearized ground effect problem of two- and three-dimensional wings is solved, using the method of matched asymptotic expansions as applied by Widnall and Barrows[2,3] to ram wings without jet flap. An important feature of the present method is that this problem is a direct problem, involving a known source distribution on the upper surface of the wing and jet sheet with concentrated sources around the leading and side edges plus a separate confined channel flow region under the wing and jet sheet. Thus, even for a general airfoil, the difficulties of inverting an integral equation are avoided. Clearly, special expansions are required in the flow region near the trailing edge which are different from those of Ref. 2, because the Kutta Joukowski hypothesis at the trailing edge is not imposed for the jet-flapped wing and the boundary condition that the jet emerges from this edge at a deflection angle is imposed at this point.

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1. Basic Formulations

The finite jet-flapped wing in close proximity to the ground in an incompressible flow is sketched in Fig. 1. The thin jet emerges from the trailing edge AA′ at initial deflection angle and this investigation of the jet-flapped wing will be restricted to the supercritical flow problem where the trailing edge jet flap does not impinge upon the ground. The ratio of the ground clearance to the root chord is defined as ε. There are two possible versions of this problem for the region beneath the wing and jet sheet, the linear and nonlinear problems. In the linear problem, the displacements of the under surfaces of the wing and jet sheet must be small in comparison to the clearance to chord ratio. This version is compatible with linearized lifting surface theory in which the lowest order solution everywhere is a free stream and the upwash boundary conditions are satisfied on the mean plane of the wing and jet sheet, at a height ε above the ground. In the nonlinear problem, the displacements of the under surface of the wing and jet sheet are of the order of the clearance. The lowest order solution for the flow beneath the wing and jet sheet must be satisfied on the actual lower surface. This latter problem is more difficult and is currently under investigation for the ram wing without jet flap. The linear problem will be examined in this paper and the results on the two-dimensional case will be compared with the linear theory by Lissaman.[1]

The wing and jet surfaces are described by $S(x, y, z) = z - \varepsilon - \alpha g_w = 0; 0 < x < 1, z - \varepsilon - \alpha g_w|_{x=1} - \alpha g_{j\alpha} - \tau g_{j\tau} = 0; 1 < x$, where α, τ are the angle of attack and jet deflection with respect to the wing root chord, g_w, $g_{j\alpha,\tau}$ are functions of x and y of $0(1)$ describing the distribution of camber on the wing and the configuration of the jet sheet. x, y, z are coordinates normalized by the mid-chord of the wing. The velocity potential Φ normalized by the uniform flow U_∞ satisfies Laplace's equation

$$\nabla^2\Phi = 0 \tag{1}$$

The boundary condition of flow tangency is

$$\nabla\Phi\cdot\nabla S = 0 \quad \text{on} \quad S = 0 \tag{2}$$

On the ground we require

$$\partial\Phi/\partial z = 0 \quad \text{for} \quad z = 0 \tag{3}$$

The downward directed jet from a wing at its trailing edge is deflected by the oncoming flow. The resulting curvature in the jet sheet results in a centrifugal force which is balanced by the difference in static pressure between the upper and lower side of the jet. Assuming that the total pressure of the outside flow above and below the jet is equal, there must obviously be a difference in velocity between the two sides of the jet sheet. This difference becomes[12]

$$(\partial\Phi/\partial x)_+ - (\partial\Phi/\partial x)_- = \tfrac{1}{2}C_\mu\,\partial^2(\alpha g_{j\alpha} + \tau g_{j\tau})/\partial x^2 \quad \text{on} \quad S = 0, \quad x > 1, \tag{4}$$

where

$$C_\mu(y) = J(y)/(\tfrac{1}{2}\rho U_\infty^2 \times 1), \qquad C_j = \int_{-b/a}^{b/a} C_\mu\, dy/S_w$$

(= jet momentum coefficient). S_w is the wing area, J is the momentum distribution of the jet at the trailing edge, a is the root chord of the wing and b is its semispan. The suffix + or − denotes the upper or lower side of the jet sheet. $g_{j\alpha,\tau}$ are, at this stage, unknown functions, but they are determined by Eq. (4) and the following conditions:

$$\partial g_{j\alpha}/\partial x = \partial g_{j\tau}/\partial x = -1, \; g_{j\alpha} = g_{j\tau} = 0 \quad \text{for} \quad x = 1 \tag{5}$$

These relations are given from the condition that the initial deflection angle of the jet at the trailing edge becomes τ.

For a jet-flapped wing at angle of attack α and angle of jet deflection τ, the vertical flow perturbations are of $0(\alpha, \tau)$, while in the confined region (i.e., the channel flow region, preserving the notation of Widnall and Barrows[2]) under the wing and jet sheet, the vertical velocity of $0(\alpha, \tau)$ induces a horizontal velocity of $0(\alpha/\varepsilon, \tau/\varepsilon)$. Hence, it is assumed in the present linear theory that the orders of α and τ are smaller than that of ε. This assumption will be valid for $\varepsilon \leq 0.1$ from the work of Widnall and Barrows[2] on a ram wing without jet flap.

In the case of the two-dimensional, jet-flapped airfoil, if the configuration of the jet sheet is determined, the one-dimensional nature of the lowest order solution for the channel flow becomes immediately apparent, but there is obviously a region near the leading edge (edge flow) where the flow reverts to a two-dimensional state. For this region, the expansion given by Widnall and Barrows[2] is required, and this solution is matched to the channel flow solution. Near the trailing edge (trailing edge flow), a special expansion is required in order to satisfy the boundary condition [Eq. (5)]. In this region, the vertical velocity of $0(\alpha, \tau)$ induces a horizontal velocity of $0(\alpha/\varepsilon, \tau/\varepsilon)$ and, moreover, the derivative of the vertical velocity with respect to x for $x > 1$ must be of $0(\alpha/\varepsilon, \tau/\varepsilon)$ from Eq. (4). From the limit matching principle, it is shown that the corresponding outer flow (away from the ground) can be represented by a distribution of sources and sinks along the ground surface. It also turns out that a concentrated source is required at the leading edge, representing the upward deflection of the stagnation streamline.

These methods are extended to three dimensions. In this case, the flow near the wing and jet sheet, which we shall persist in calling the channel flow, is two-dimensional, and the leading edge flow solution may be applied along the side edges of the wing and jet sheet. In the outer region, the flow can be represented by a distribution of sources and sinks on the wing and jet sheet plus a distribution of concentrated sources around the leading edge and side edges, whose strength is determined by matching. And the configuration of the jet is determined from Eqs. (4) and (5). From the procedure of the analysis, it will be easily seen that the present theory is valid for the aspect ratio $AR > 0(\varepsilon)$.

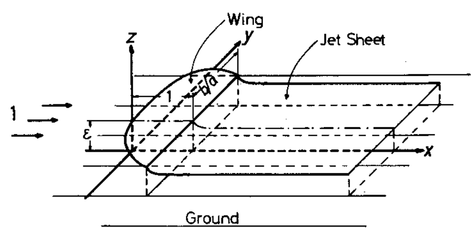


Fig. 1 The flow around a finite jet-flapped wing in close proximity to the ground.

1.1 Channel Flow Solution

To determine a perturbation solution valid for the region near the wing and jet sheet where z is of $0(\varepsilon)$, the z co-ordinate is stretched as $z_i = z/\varepsilon$ and other independent variables x, y are left unstretched. When $\phi^c(x, y, z_i)$ is the perturbation potential in this channel flow region ($\Phi = x + \phi^c$), an asymptotic expansion of the form $\phi^c = \varepsilon^{-1}\phi_0^c + \varepsilon\phi_1^c + \varepsilon^3\phi_2^c + \cdots$ will be assumed, where $0(\phi_0^c) > 0(\varepsilon^2)$, $0(1) \geq 0(\phi_i^c) > 0(\varepsilon^2)$; $i = 1, 2, \ldots$. Substituting this expansion into the governing Eq. (1)

$$\phi_0^c = \zeta_0^c z_i + \psi_0^c, \quad \phi_1^c = -(1/3!)\nabla_{2D}^2\zeta_0^c z_i^3 - (1/2!)\nabla_{2D}^2\psi_0^c z_i^2 + \zeta_1^c z_i + \psi_1^c, \ldots$$

where ζ_n^c and ψ_n^c ($n = 0, 1, \ldots$) are, at this stage, some arbitrary functions of x and y, $\nabla_{2D} = \mathbf{i}\partial/\partial x + \mathbf{j}\partial/\partial y$; $\mathbf{i}, \mathbf{j}$ are unit vectors with respect to x, y. Since $\alpha, \tau \sim o(\varepsilon)$, a Taylor series expansion of the boundary condition (2) about $z_i = 1$ is permitted and only terms linear in α and τ would be appropriate in the present linear theory. Therefore, $\zeta_0^c = 0$, $\zeta_1^c - \nabla_{2D}^2\psi_0^c = \alpha\partial g_w/\partial x$; $0 < x < 1$, $\zeta_1^c - \nabla_{2D}^2\psi_0^c = \alpha\partial g_{j\alpha}^0/\partial x + \tau\partial g_{j\tau}^0/\partial x$; $1 < x$ where $\partial g_{j\alpha,\tau}/\partial x = \partial g_{j\alpha,\tau}^0/\partial x + 0(\varepsilon^2)$. Also from the ground condition (3), $\zeta_1^c = 0$.

The flowfield beneath the wing and jet sheet is required to satisfy the ground condition (3), but the flowfield above the wing and jet sheet is not required.

$$\phi^c_+ = \varepsilon^{-1}\psi^c_{0+} + \varepsilon[(z_i - z_i^2/2)\nabla^2_{2D}\psi^c_{0+} + \psi^c_{1+} + z_i\alpha\partial g_w/\partial x] + 0(\varepsilon^3) \quad \text{for} \quad 0 < x < 1$$
$$= \varepsilon^{-1}\psi^c_{0+} + \varepsilon[(z_i - z_i^2/2)\nabla^2_{2D}\psi^c_{0+} + \psi^c_{1+} + z_i(\alpha\partial g^0_{j\alpha}/\partial x + \tau\partial g^0_{j\tau}/\partial x)] + 0(\varepsilon^3) \quad \text{for} \quad 1 < x \tag{6}$$

$$\phi^c_- = \varepsilon^{-1}\psi^c_{0-} + \varepsilon[\psi^c_{1-} + (z_i^2/2)\alpha\partial g_w/\partial x] + 0(\varepsilon^3) \quad \text{for} \quad 0 < x < 1$$
$$= \varepsilon^{-1}\psi^c_{0-} + \varepsilon[\psi^c_{1-} + (z_i^2/2)(\alpha\partial g^0_{j\alpha}/\partial x + \tau\partial g^0_{j\tau}/\partial x)] + 0(\varepsilon^3) \quad \text{for} \quad 1 < x \tag{7}$$

where

$$\nabla^2_{2D}\psi^c_{0-} = -\alpha\partial g_w/\partial x; \quad 0 < x < 1, \quad -(\alpha\partial g_{j\alpha}/\partial x + \tau\partial g_{j\tau}/\partial x); \quad 1 < x \tag{8}$$

And suffix + or − denotes the flowfield above or below the wing and jet sheet. When we consider only terms linear in α and τ in the jet sheet condition (4)

$$\partial\psi^c_{0+}/\partial x - \partial\psi^c_{0-}/\partial x = (\varepsilon C_\mu/2)\partial^2(\alpha g^0_{j\alpha} + \tau g^0_{j\tau})/\partial x^2; \quad 1 < x \tag{9}$$

1.2 Trailing Edge Flow Solution

In this region, as mentioned in Sec. 1, the derivative of the vertical velocity with respect to x for $x > 1$ must be of the order of the horizontal velocity of $0(\alpha/\varepsilon, \tau/\varepsilon)$. When the x coordinate is stretched as $x_t = \varepsilon^{-1/2}(x-1)$, the vertical velocity of $0(\alpha, \tau)$ induces a horizontal velocity of $0(\alpha/\varepsilon^{1/2}, \tau/\varepsilon^{1/2})$ and the derivative of vertical velocity with respect to x for $x_t > 0$ becomes of $0(\alpha/\varepsilon^{1/2}, \tau/\varepsilon^{1/2})$. Hence, the independent variables are taken as x_t, y, z_i in this trailing edge flow region. Since the horizontal velocity is of $0(\alpha/\varepsilon^{1/2}, \tau/\varepsilon^{1/2})$ in this region, $\phi^t(x_t, y, z_i)$ defined as the perturbation potential in this region will be assumed to be expanded in the form $\phi^t = \phi^t_0 + \varepsilon\phi^t_1 + \varepsilon^2\phi^t_2 + \cdots$ where $0(\phi^t_0) > 0(\varepsilon)$, $0(1) \geq 0(\phi^t_i) > 0(\varepsilon)$; $i = 1, 2, \ldots$. As mentioned in Sec. 1.1, substituting this expansion into Eq. (1) and considering the flow tangency conditions (2) and (3)

$$\phi^t_+ = \psi^t_{0+} + \varepsilon[(z_i - z_i^2/2)\partial^2\psi^t_{0+}/\partial x_t^2 + \psi^t_{1+} + z_i(\alpha\partial g_w/\partial x|_{x=1} + \alpha\varepsilon^{1/2}x_t\partial^2 g_w/\partial x^2|_{x=1})] + 0(\varepsilon^2); \quad x_t < 0$$
$$= \psi^t_{0+} + \varepsilon[(z_i - z_i^2/2)\partial^2\psi^t_{0+}/\partial x_t^2 + \psi^t_{1+} + z_i(\alpha G^0_{j\alpha} + \tau G^0_{j\tau})] + 0(\varepsilon^2); \quad x_t > 0 \tag{10}$$

$$\phi^t_- = \psi^t_{0-} + \varepsilon[\psi^t_{1-} + (z_i^2/2)(\alpha\partial g_w/\partial x|_{x=1} + \alpha\varepsilon^{1/2}x_t\partial^2 g_w/\partial x^2|_{x=1})] + 0(\varepsilon^2); \quad x_t < 0$$
$$= \psi^t_{0-} + \varepsilon[\psi^t_{1-} + (z_i^2/2)(\alpha G^0_{j\alpha} + \tau G^0_{j\tau})] + 0(\varepsilon^2); \quad x_t > 0 \tag{11}$$

where

$$\partial g_{j\alpha,\tau}/\partial x|_{x=1+\varepsilon^{1/2}x_t} = G^0_{j\alpha,\tau}(x_t, y) + 0(\varepsilon)$$
$$\partial^2\psi^t_{0-}/\partial x_t^2 = -\alpha[\partial g_w/\partial x|_{x=1} + \varepsilon^{1/2}x_t\partial^2 g_w/\partial x^2|_{x=1}]; \quad x_t < 0$$
$$= -(\alpha G^0_{j\alpha} + \tau G^0_{j\tau}); \; x_t > 0 \tag{12}$$

and $\psi^t_{0\pm}$, $\psi^t_{1\pm}$ are the functions of x_t and y. From the jet sheet condition (4)

$$\partial\phi^t_+/\partial x_t - \partial\phi^t_-/\partial x_t = (C_\mu/2)\partial(\alpha G^0_{j\alpha} + \tau G^0_{j\tau})/\partial x_t; \quad x_t > 0, \quad z_i = 1 \tag{13}$$

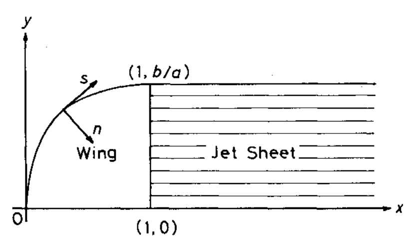


Fig. 2 The Cartesian coordinates in the edge flow.

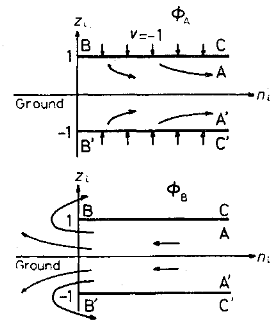


Fig. 3 The perturbation velocity potential ϕ_A and ϕ_B; ϕ_A satisfies the downwash condition $\partial\phi_A/\partial z_i = -1$, ϕ_B is the eigensolution with homogeneous boundary condition.

1.3 Edge Flow Solution

As already studied by Widnall and Barrows,[2] the flow near the edge of the wing and jet sheet is considered to be different from the trailing edge flow or the channel flow. In this region, Cartesian coordinates are as shown in Fig. 2.

$\phi^i(s, n_i, z_i)$ is the perturbation potential in this edge flow region and n_i are taken as $n_i = n/\varepsilon$. The function ϕ^i will be assumed to be expanded as the following form $\phi^i = \phi^i_0 + \varepsilon^2\phi^i_1 + \varepsilon^4\phi^i_2 + \cdots$ where $0(\phi^i_0) > 0(\varepsilon^2)$, $0(1) \geq 0(\phi^i_j) > 0(\varepsilon^2)$; $j = 1, 2, \ldots$. When we substitute this expansion into Eq. (1) and consider Eqs. (2) and (3), the edge flow ϕ^i can be easily obtained in the form

$$\phi^i = -\varepsilon\alpha g_{wx}(s)\phi_A + \phi_B + A(s)n_i + B(s) + 0(\varepsilon^2) \quad \text{for} \quad 0 < x(s) < 1$$
$$= -\varepsilon[\alpha g^0_{jx\alpha}(s) + \tau g^0_{jx\tau}(s)]\phi_A + \phi_B + A(s)n_i + B(s) + 0(\varepsilon^2) \quad \text{for} \quad 1 < x(s) \tag{14}$$

where the position of the edge of the wing or the jet sheet is $[x(s), y(s), \varepsilon]$ and $\partial g_w/\partial x$, $\partial g^0_{j\alpha,\tau}/\partial x$ for $x = x(s)$ and $y = y(s)$ are defined as $g_{wx}(s)$, $g^0_{jx\alpha,\tau}(s)$, and ϕ_A, ϕ_B are functions of n_i and z_i, ϕ_A is the solution which satisfies the downwash condition $\partial\phi_A/\partial z_i = -1$ on the wing or on the jet sheet, and ϕ_B is the eigensolution with homogeneous boundary condition; i.e., no velocity normal to the wing or the jet sheet (see Fig. 3). A, B are unknown functions of s to be determined by matching.

Using conformal transformations, the following relations are obtained:

$$Z = n_i + iz_i = (\zeta^2 - 1 - 2\log\zeta)/\pi + i$$
$$F_B = \phi_B + i\psi_B = [2U(s)/\pi]\log\zeta - iU(s)$$
$$\partial F_A/\partial Z = \partial\phi_A/\partial n_i - i\partial\phi_A/\partial z_i = -(2/\pi)\log\zeta + i$$

where U is an unknown function of s. For matching, it is necessary to invert this relation for the asymptotic cases of large n_i

$$F_{B+} \simeq (U/\pi)\log\pi Z + iU, \quad F_{B-} \simeq -U(1/\pi + Z)$$
$$F_{A+} \simeq -(Z/\pi)\log\pi Z + Z/\pi + iZ, \quad F_{A-} \simeq (1/\pi + Z/2)Z \tag{15}$$

1.4 Outer Flow Solution

In the outer flow region, the independent variables are taken as x, y, z, and the perturbation potential $\phi^0(x, y, z)$ in this region must be matched to the channel flow potential and to the edge flow potentials. The asymptotic expansion of ϕ^0 will be assumed as $\phi^0 = \phi^0_0 + \varepsilon\phi^0_1 + \varepsilon^2\phi^0_2 + \cdots$. Since $\phi^0_0, \phi^0_1, \ldots$ can be expanded as a Taylor series about $z = 0$, we compare with Eqs. (6) and (10), in accordance with the limit matching principle of Van Dyke[14]

$$\psi^c_{0+} = \varepsilon\phi^0_0(x, y, 0), \quad \psi^c_{1+} = \phi^0_1(x, y, 0) + 0(\varepsilon)$$
$$\partial\phi^0_0/\partial z|_{z=0} = \alpha\partial g_w/\partial x; \; 0 < x < 1, \; \partial(\alpha g^0_{j\alpha} + \tau g^0_{j\tau})/\partial x; \; 1 < x \tag{16}$$

and

$$\psi^t_{0+} \simeq \phi^0_0(1 + \varepsilon^{1/2}x_t, y, 0), \quad 0(\partial^2\psi^t_{0+}/\partial x_t^2) < 0(1)$$
$$(\partial\phi^0_0/\partial z)_{x=1+\varepsilon^{1/2}x_t, y=y, z=0} = \alpha(\partial g_w/\partial x|_{x=1} + \varepsilon^{1/2}x_t\partial^2 g_w/\partial x^2|_{x=1}) + 0(\varepsilon); \; x_t < 0$$
$$= \alpha G^0_{j\alpha} + \tau G^0_{j\tau} + 0(\varepsilon); \quad x_t > 0 \tag{17}$$

From Eq. (15), $\phi_0^0 \sim (U/\pi)\log|Z|;\ x, y \to x(s), y(s), z \to 0$. Hence, it is easily seen from these relations that the downwash condition is satisfied by a distribution of sources (or sinks) and the concentrated sources at the leading and side edges must be added to the outer flow. This concentrated source strength U is chosen to satisfy mass conservation.

$$\phi_0^0 = -\frac{1}{2\pi}\iint_{Sw}\frac{\alpha}{r}\frac{\partial}{\partial\xi}g_w(\xi,\eta)d\xi\, d\eta - \frac{1}{2\pi}\iint_{Sj}\frac{1}{r}\frac{\partial}{\partial\xi}[\alpha g_{j\alpha}(\xi,\eta) + \tau g_{j\tau}(\xi,\eta)]d\xi\, d\eta - \frac{1}{2\pi}\int_C\frac{U(s)}{r_\varepsilon}ds \tag{18}$$

where $r^2 = (x-\xi)^2 + (y-\eta)^2 + z^2$, $r_\varepsilon^2 = [x - x(s)]^2 + [y - y(s)]^2 + z^2$, and S_j is the area of the jet sheet.

2. Jet Sheet Configuration

The function ψ_{0-}^c for $x > 1$ is obtained from Eqs. (9) and (16) as $\psi_{0-}^c = D_0(y) + 0(\varepsilon)$. Substituting this equation into Eq. (8), $\partial(\alpha g_{j\alpha}^0 + \tau g_{j\tau}^0)/\partial x = -d^2D_0/dy^2$. And when we substitute Eq. (13) into Eq. (12) and consider Eq. (17)

$$\alpha G_{j\alpha}^0 + \tau G_{j\tau}^0 - (C_\mu/2)\partial^2(\alpha G_{j\alpha}^0 + \tau G_{j\tau}^0)/\partial x_t^2 = 0(\varepsilon^{1/2}) \tag{19}$$

From the initial condition (5)

$$G_{j\alpha}^0 = 0(\varepsilon^{1/2}), \quad G_{j\tau}^0 = -\exp(-kx_t) + 0(\varepsilon^{1/2}), \quad k = (2/C_\mu)^{1/2}$$

From the limit matching principle between $g_{j\alpha,\tau}$ and $G_{j\alpha,\tau}$ the composite solution of the configuration of the jet sheet becomes

$$g_j(x,y) = (\varepsilon^{1/2}\tau/k)\{\exp[-k\varepsilon^{-1/2}(x-1)] - 1\} + 0(\varepsilon) \tag{20}$$

It is interesting that the configuration of the jet in the lowest order solution is independent of α.

3. Matching Procedure

At this section, we try to match the various regions of the flow in accordance with the limit matching principle by Van Dyke.[14] In the trailing edge region, when we consider that the velocity potential and velocity are continuous at the trailing edge and we compare ϕ_-^{tc} with ϕ_-^{ct} (i.e., these notations preserve those of Widnall and Barrows[2]),

$$\begin{aligned}\psi_{0-}^c(1+0,y) &= \varepsilon[D(y) - \tau/k^2]\\ \partial\psi_{0-}^c/\partial x|_{x=1+0} &= \varepsilon^{1/2}[C(y) + \tau/k]\\ \partial^2\psi_{0-}^c/\partial x^2|_{x=1+0} &= \partial^3\psi_{0-}^c/\partial x^3|_{x=1+0} = 0\end{aligned} \tag{21}$$

and

$$\begin{aligned}\psi_{0-}^c(1-0,y) = \varepsilon D(y), \quad \partial\psi_{0-}^c/\partial x|_{x=1-0} &= \varepsilon^{1/2}C(y)\\ \partial^2\psi_{0-}^c/\partial x^2|_{x=1-0} &= -\alpha\partial g_w/\partial x|_{x=1}\\ \partial^3\psi_{0-}^c/\partial x^3|_{x=1-0} &= -\alpha\partial^2 g_w/\partial x^2|_{x=1}\end{aligned} \tag{22}$$

where C and D are unknown functions of y. The last two equations

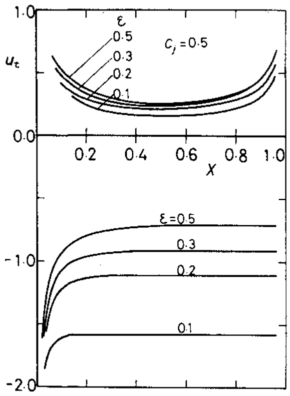


Fig. 4 The perturbation velocity for $\alpha = 0$ along the upper and lower surfaces of the airfoil.

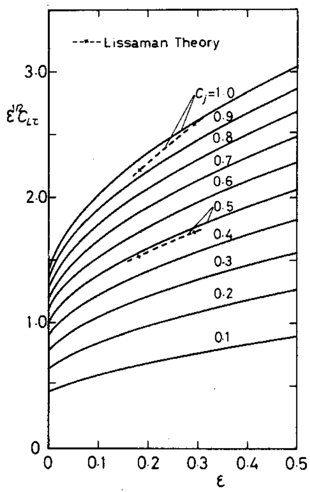


Fig. 5 The effect of the lift derivative $C_{L\tau}$ for the ground clearance ε.

of these relations may be satisfied identically from Eq. (8). The following relations are given by $\psi_{0-}^c \cong D_0(y) + 0(\varepsilon)$

$$\partial\psi_{0-}^c/\partial x|_{x=1-0} = -\tau\varepsilon^{1/2}/k + 0(\varepsilon), \quad \psi_{0-}^c(1-0,y) = \varepsilon D(y) \tag{23}$$

Moreover,

$$\phi^{0i} \cong (U/\pi)\log\varepsilon n_i + \varepsilon(n_i/\pi)\log\varepsilon n_i g_{wx}(s) + \varepsilon A^0(s)n_i + B^0(s) + 0(\varepsilon); \quad 0 < x(s) < 1$$

where A^0 and B^0 are the functions of s induced from Eq. (18). From ϕ^{i0} induced from Eq. (15) and ϕ^{0i}, $B(s) = B^0 - (U/\pi)\log(\pi/\varepsilon)$, $A(s) = 0(\varepsilon)$. Comparing ϕ_-^{ic} induced from Eq. (15) with ϕ_-^{ci} from Eq. (7)

$$\psi_{0-}^c[x(s), y(s)] = \varepsilon(B - U/\pi), \quad \partial\psi_{0-}^c/\partial n|_{x=x(s),y=y(s)} = -U + 0(\varepsilon)$$

From these relations, the lowest order solution of ψ_{0-}^c is obtained from Eq. (8)

$$\psi_{0-}^c[x(s), y(s)] = 0, \quad \partial\psi_{0-}^c/\partial x|_{x=1-0} = -\tau\varepsilon^{1/2}/k \tag{24}$$

The circulation $\Gamma(y)$ around the wing is given as

$$\Gamma(y) = \phi_+^t(1-0,y) - \phi_-^t(1-0,y) = -D(y) + 0(1)$$

And the lift coefficient C_L is also given

$$C_L = \frac{2}{S_w}\int_{-b/a}^{b/a}\Gamma(y)dy + \frac{1}{S_w}\int_{-b/a}^{b/a}(\alpha + \tau)C_\mu(y)dy$$

Since the linear theory is examined in this paper, it is easily seen, from Eq. (20), that the lowest order solution for zero angle of jet deflection is identical with the solution given by Widnall and Barrows.[2] Hence, the present paper deals primarily with the problem for the case of the zero angle of attack. Also, for a two-dimensional jet-flapped flat plate wing, the relation between $\partial C_L/\partial\alpha(\equiv C_{L\alpha})$ and $\partial C_L/\partial\tau(\equiv C_{L\tau})$ was already given by Lissaman[1] as

$$C_{L\tau}^2 = 2C_jC_{L\alpha} - C_j^2 \tag{25}$$

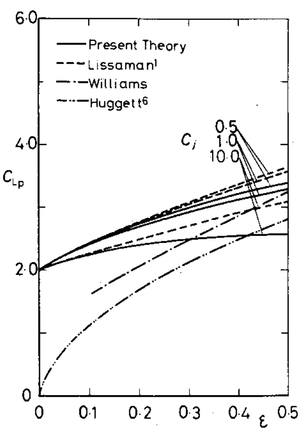


Fig. 6 The blockage curve in the case of $\alpha = 0$.

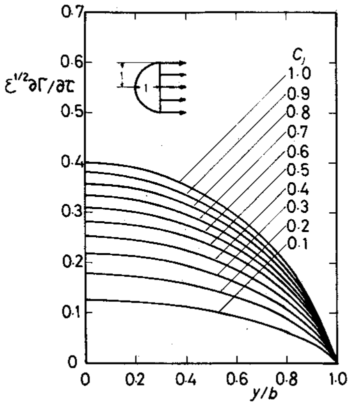


Fig. 7 The spanwise distribution of the circulation about a semicircular flat plate wing with a uniform jet in the case of $\alpha = 0$.

3.1 The Two-Dimensional Wing in Ground Effect

In two-dimensional case, the channel flow is easily solved as a one-dimensional problem, and the rule of the matching discussed in the previous section is applied to this problem. When we consider the third-order approximate solution, the lift derivative $C_{L\tau}$ is obtained as

$$C_{L\tau} = C_j + 2[\varepsilon^{-1/2}/k - 3\varepsilon^{1/2}\log\varepsilon/(2\pi k) + \varepsilon^{1/2}(\gamma + \log k + 1 + \log\pi)/(\pi k)] + 0(\varepsilon), \quad k = (2/C_j)^{1/2}$$

Using this equation and Eq. (25), $C_{L\alpha}$ is also obtained, but the term of the order of $0(\varepsilon^{1/2})$ of $C_{L\tau}$ is not completely correct and this term will be correctly obtained by considering the higher approximation of Eq. (19).

The perturbation velocity u_τ for $\alpha = 0$ along the upper and lower surfaces of the airfoil is shown in Fig. 4, where the velocity U_l along these surfaces is given as $U_l = 1 + \tau u_\tau$. This velocity is obtained by considering the composite perturbation potential ϕ^{com} $(= \phi^c + \phi^t + \phi^i - \phi^{tc} - \phi^{ic})$.

Here, we consider the blockage curve. The wake blockage occurs when the jet impinges on the floor. Hence, the limiting condition of the blockage becomes $\tau/k = \varepsilon^{1/2}$ from Eq. (20). The pressure force C_{Lp} for this limiting case and $\alpha = 0$ is obtained as

$$C_{Lp} = 2\{1 - 3\varepsilon\log\varepsilon/(2\pi) + \varepsilon[\gamma + \log(\tau/\varepsilon^{1/2}) + \log\pi]/\pi\} + 0(\varepsilon^2) \qquad (26)$$

The effect of the lift derivative $C_{L\tau}$ for the ground clearance is shown in Fig. 5. From this figure, the present method shows reasonable agreement for $\varepsilon \leq 0.5$, comparing with the results of Lissaman[1] (i.e., the calculated coefficients lie within 5% of corresponding his results). Figure 6 shows the blockage curve given by Eq. (26). In this figure, the results of Huggett,[6] Lissaman[1] and Williams (here, the results of Williams are plotted from the work of Lissaman[1]) are represented. For the blockage predictions of Huggett and Williams C_{Lp} should represent the maximum pressure lift carried by the airfoil. For that of Lissaman, it corresponds to the C_{Lp} at which there is no net flow under the airfoil, but for the present theory it corresponds to the C_{Lp} at which the jet impinges upon the floor at infinity. From these curves, it will be seen that the present blockage prediction is in fairly reasonable correlation with that of Lissaman.

3.2 Three-Dimensional Wing in Ground Effect

Since the lowest order solution for the angle of attack is identical with that of Widnall and Barrows,[2] it is obtained in this section for a flat plate semielliptic wing with a straight trailing edge in the case of the uniform jet distribution and zero angle of attack.

3.2.1 *Semielliptic planform*

The function which conformally transforms the interior of ellipse (z plane) on the interior of circular (ζ plane) is

$$z = (A^2 - B^2)^{1/2}\sin\left\{\frac{\pi}{2K}\int_0^{\zeta/k_1^{1/2}} \frac{dx}{[(1 - x^2)(1 - k_1^2x^2)]^{1/2}}\right\}$$

where A and B are semiaxis of ellipse ($A \geq B$). Therefore, the function ψ_{0-}^{ξ} is given

$$\psi_{0-}^{\xi}(x, y) = \frac{\varepsilon^{1/2}}{2\pi k}\int_{-1}^{1} \log\left[\frac{\xi^2 + (\eta - t)^2}{t^2\xi^2 + (t\eta - 1)^2}\right]\frac{dy}{d\eta}\bigg|_{\eta=t} dt \qquad (27)$$

where

$$z = x + iy, \quad \zeta = \xi + i\eta, \quad K = F(k_1, \pi/2)$$

$$e^{i\pi\tau} = [(A - B)/(A + B)]^2, \quad k_1 = [\Theta_2(\tau)/\Theta_3(\tau)]^2$$

$$\Theta_2(\tau) = \vartheta_2(0|\tau), \quad \Theta_3 = \vartheta_3(0|\tau)$$

Here, F and $\vartheta_{2,3}$ are the elliptic integral of the first kind and the ϑ function of the second, third kind.

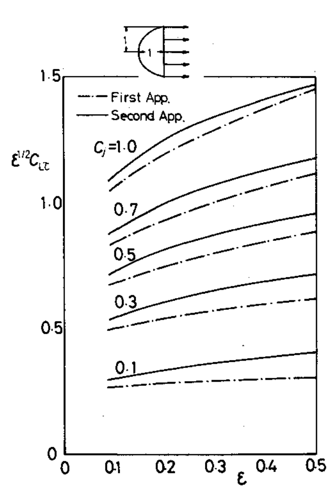
$$C_{L\tau} = C_j - \frac{8\varepsilon^{-1/2}}{\pi^2 ABk}\int_0^1\int_0^1 \log\left(\frac{\eta^2 - t^2}{\eta^2t^2 - 1}\right)\frac{dy}{d\eta}\frac{dy}{d\eta}\bigg|_{\eta=t} dt\,d\eta \qquad (28)$$

where $C_\mu = \pi C_j/4$, $k = (2/C_\mu)^{1/2}$.

For the semicircular planform case, the spanwise distribution of the circulation $\Gamma(y)$ is shown in Fig. 7, and the lift derivative $C_{L\tau}$ of the lowest order solution and of the two-term solution expanded to $0(\varepsilon^{1/2})$ are shown in Fig. 8, but it is assumed that the terms of $0(\varepsilon^{1/2})$ in Eq. (19) may be neglected in the two-term solution.

For the semielliptic planform case, from Eq. (28), the values of $(C_{L\tau} - C_j)(\varepsilon/C_j)^{1/2}$ are independent of C_j and ε. Hence, these values are shown in Fig. 9.

Fig. 8 The lift coefficient $C_{L\tau}$ of the first and second-order solutions on a semicircular flat plate wing with a uniform jet in the case of $\alpha = 0$.



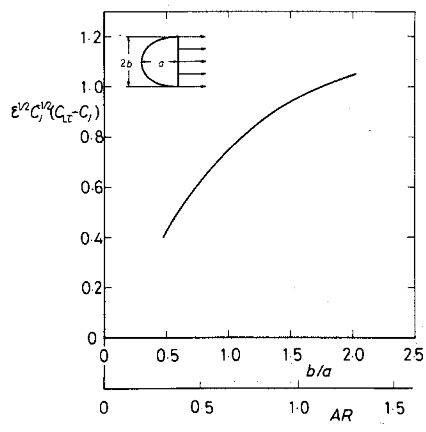


Fig. 9 The lift coefficient $C_{L\tau}$ of the lowest order solution on a semielliptic flat plate wing with a uniform jet in the case of $\alpha = 0$.

3.2.2 *Induced drag*

In the ground effect problem, the induced drag coefficient C_{Df} induced by the free vortex sheet is

$$C_{Df} = -\int_{-b/a}^{b/a} \Gamma_\infty w_\infty \, dy/S_w$$

and the drag of the jet becomes

$$\tfrac{1}{2}\int_{-b/a}^{b/a} J(y) w_\infty^2 \, dy$$

where Γ_∞ is the circulation about the wing and jet sheet and w_∞ is the downwash velocity at infinity; $w_\infty = \partial(\alpha g_{j\alpha} + \tau g_{j\tau})/\partial x|_{x\to\infty}$. Here, since $w_\infty \sim 0(\varepsilon^{1/2})$ from Eq. (27) and C_{Df} of the lowest order solution is of 0(1), the induced drag coefficient C_{Di} becomes

$$C_{Di} \cong 2\varepsilon \int_{-b/a}^{b/a} (\partial\Gamma/\partial y)^2 dy/S_w + 0(\varepsilon^{1/2})$$

For a semicircular planform,

$$C_{Di} = 8(\pi^2 - 6)/(9\pi k^2) + 0(\varepsilon^{1/2})$$

Widnall and Barrows[2] indicate that the semielliptic planform without jet flap is optimal for a wing in ground effect for all aspect ratios, for this wing has minimum drag for a given lift. Hence, it may be important to determine the distribution of the jet momentum at the trailing edge with which the semielliptic wing has minimum drag for a given lift. When the induced downwash is constant, this wing has minimum drag for a given lift.[3] Therefore, since $\Gamma \sim 0(\varepsilon^{-1/2})$, $\Gamma(y) = c_0 \varepsilon^{-1/2}(y - b/a)(y + b/a)$. The distribution of the jet momentum is obtained as

$$\frac{1}{k} = \left(\frac{C_\mu}{2}\right.^{1/2} = \pi c_0 \; y - \frac{b}{a} y + \frac{b}{a} \int_0^1 \log\left(\frac{\eta^2 - t^2}{\eta^2 t^2 - 1} \; \frac{dy}{d\eta}\right._{\eta = t} dt$$

Therefore, for the semicircular wing,

$$C_\mu(y) = 2[\pi c_0 y/\log(1 + y)/(1 - y)]^2$$

4. Conclusions and Discussion

A linearized lifting surface problem for a jet-flapped wing very close to the ground has been formulated using the method of matched asymptotic expansions. By using this method, the lifting problem close to the ground can be solved as a direct problem involving a source-sink rather than a vortex distribution. The flow in the confined region beneath the wing and jet sheet is a two-dimensional channel flow with known boundaries and mass addition, coming from the boundary condition of the flow tangency. The lowest order solutions are, of course, linear in α and τ, and the solution for $\tau = 0$ is identical with that found by Widnall and Barrows[2] on the wing without jet flap. So, in the present paper, the lowest order solutions on $\alpha = 0$ are obtained for semielliptic planforms and this method can be easily applied to any planforms of the jet-flapped wing. The jet momentum distribution at the trailing edge for minimum induced drag is given for the semielliptic wing which, without jet flap, has minimum induced drag.

The two-dimensional linearized flow problem for a lifting flat plate close to the ground can be solved easily by this method. The analytic solution is obtained up to $0(\varepsilon^{1/2})$. Comparison with the results of Lissaman[1] is carried out and it is seen that these results show fairly good agreement with his results for clearance $\varepsilon \leq 0.5$. The singularity $0(1/x)$ at the leading edge shown by the flat plate outer solution can be removed by introducing the composite solution. Moreover, the blockage curve also is obtained, and it is shown that the maximum pressure lift becomes constant for small ground clearance, in spite of the fact that the pressure lift is inversely proportional to the square of the ground clearance.

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